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Edge delamination in a laminated composite strip under generalized plane deformations

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Abstract. Edge delamination cracks in an elastic laminated composite strip are considered within the range of generalized plane deformations. By way of a simple regular finite element method, the method of mutual integral, wherein generalized plane strain solutions are employed as auxiliary fields, is applied to edge delamination cracks to obtain the complex stress intensities. The proposed numerical scheme is found to be very efficient and accurate. Moreover, the crack growth stability is examined for various loadings, including compression, bending and torsion, in terms of the energy release rate and mode mixity.

1. Introduction

Delamination is one of the major failure modes for composite materials, and there have been considerable efforts to analyze the behavior of delamination cracks from the viewpoint of fracture mechanics. Accordingly there have been numerous works along this line, just citing a few, but not limited to [1]–[6], but it is not appropriate to attempt to mention all of them.

For opened interfacial cracks, it is well known that the stress intensities are not decomposed into the three independent modes in the classical sense, but they are inherently coupled with one another because of their oscillatory nature. Suo [4] recently showed that the near tip field for interfacial cracks in an anisotropic composite body is characterized by one complex parameter and one real parameter. For accurate calculation of stress intensities for interfacial cracks in composite laminates, some special numerical techniques have been used, for example: boundary collocation method (Wang [1]), singular hybrid FEM (Wang and Yuan [6]). Recently, Suo's complex stress intensities [4] for delamination, as a special case of wedge type cross sections, in a laminated composite strip under generalized plane deformations have been calculated via hybrid FEM (Kim and Im [7]) and via enriched FEM (Jeon and Im [8]). The aforementioned special numerical methods involve considerable complexities in implementation as well as in formulation although they retain good accuracy. In this paper we are concerned with calculating Suo's complex stress intensities for delamination cracks in a laminated composite strip under the so called generalized plane deformation [9] in a more efficient way: firstly an expression for the J-integral for delamination cracks under generalized plane deformation is obtained, and next a displacement based regular FEM is employed to obtain the stress intensities with the aid of the so called mutual interaction integral, stemming from the J-integral for an elastic field obtained from superposition of two independent elastic states [10]. Numerical results are compared with the results obtained from the enriched FEM, and some remarks are made regarding the efficiency and accuracy of the present scheme.

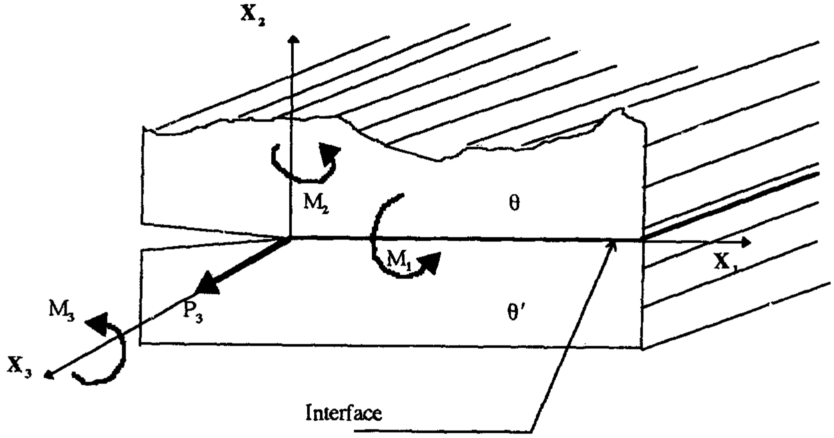

Figure 1. A delamination crack in a laminated composite strip under generic loadings yielding generalized plane deformations.

2. The statement of the problem and the basic equations

Consider a sufficiently long laminated composite strip under generalized plane deformation wherein the state of stress and strain does not vary along the generator or the axial coordinate of the strip. Suppose this strip contains an edge delamination crack the configuration of which does not change along the axial coordinate. The generalized plane deformation of the strip may then occur when it is subjected to the end loading of axial force, pure bending moment and/or torsion with no shear forces upon the lateral surface of the strip (see Figure 1). Included in the generalized plane deformation are also the generalized plane strain problems, wherein self-equilibrated lateral traction orthogonal to the generator of the strip is applied uniformly independent of the coordinate along the strip length.

Suppose an edge delamination crack exists in a laminated composite strip under generalized plane deformation, as shown in Figure 1. We take a coordinate system with origin at the crack-tip such that the x_3-axis is along the axial direction of the strip and the x_2-axis is along the thickness direction of the laminate. Then the x_1-axis is aligned along the width direction of the strip or along the ply interface of the laminate, and each ply of the composite laminate lies in a plane parallel to the x_1-x_3 plane. The ply orientation is now defined to be the counter-clockwise angle, viewed from the top, that the fiber direction makes with the x_3-axis. The orientations of the adjacent plies between which a delamination crack is formed is denoted as $[\theta/\theta']$, where θ and θ' are the ply orientation of the upper and the lower ply, respectively.

The two-dimensional formulation of such a class of deformations is tractable once we decompose the deformation into two parts: one is the cross-sectional distortion which is independent of the axial coordinate of the strip and the other is the remaining part dependent upon the axial coordinate as well as the coordinates in the cross-section. According to Lekhnitskii [9], the displacement u_i may indeed be written as

$$\begin{aligned} u_i(x_1, x_2, x_3) &= U_i(x_1, x_2) + \delta_{i1}\left(-\frac{A_2}{2}x_3^2 - A_4 x_2 x_3\right) \\ &+\delta_{i2}\left(-\frac{A_3}{2}x_3^2 + A_4 x_1 x_3\right) + \delta_{i3}(A_2 x_1 + A_3 x_2 + A_1)x_3, \end{aligned} \tag{2.1}$$

where δ_{ij} is the Kronecker delta, and A_1 is a deformation parameter related to axial extension along the x_3-axis; A_2 and A_3 parameters related to curvature in the x_1-x_3 planes and the x_2-x_3 planes, respectively, and A_4 is related to twist along the x_3-axis. Let ε_{ij} and σ_{ij} denote the Cartesian components of strain and stress tensors, respectively. For the aforementioned class of deformations, we have the following governing equations:

$$\sigma_{i\beta,\beta} = 0 \quad (i = 1,2,3, \quad \beta = 1,2), \tag{2.2}$$

$$\varepsilon_{ij} = \frac{(u_{i,j} + u_{j,i})}{2}, \tag{2.3}$$

$$\sigma_{ij} = C_{ijkm}\varepsilon_{km}, \qquad C_{ijkm} = C_{jikm} = C_{kmij}, \tag{2.4}$$

where C_{ijkm} is a 4-th order stiffness tensor, and the comma indicates the partial differentiation with respect to x_i. Note that the aforementioned governing equations and all relevant equations to follow, in principle, have to be written for each of the two adjacent plies near a delamination crack. We suppose the loadings of uniaxial tension, pure bending and torsion, only, which result in the state of generalized plane deformation in the composite strip, and make the problem two dimensional.

Substituting the expression for displacement (2.1) into the equilibrium equation (2.2), we obtain the following governing equation for $U(x_1, x_2)$

$$C_{i1k1}\frac{\partial^2 U_k}{\partial x_1^2} + (C_{i1k2} + C_{i2k1})\frac{\partial^2 U_k}{\partial x_1 \partial x_2} + C_{i2k2}\frac{\partial^2 U_k}{\partial x_2^2} + (C_{i123}A_4 + C_{i133}A_2) + (C_{i233}A_3 - C_{i213}A_4) = 0. \tag{2.5}$$

In the case of the end loading being prescribed on a cross-section, the deformation parameters A_i in the above equation may be determined from the condition that the traction resultants are in equilibrium with the prescribed loadings such as the axial force, the bending moment and/or the twisting moment. Note that (2.5) with the deformation parameters $A_i (i = 2,3,4)$ being zero is just the governing equation for the generalized plane strain deformation. Moreover, this is an elliptic partial differential equation, characteristics of which are complex. The asymptotic homogeneous solution for interfacial cracks, which comes out of the Williams type eigenfunction expansion [12], has been obtained in terms of the complex characteristics by Ting [3] and Kim and Im [7] as follows

$$U_i^h = \sum_{n=1}^{\infty}\sum_{k=1}^{3}\frac{[\beta_n b_{kn}\nu_{ik}z_k^{\delta_n+1} + \beta_n b_{(k+3)n}\overline{\nu}_{ik}\overline{z}_k^{\delta_n+1}]}{(\delta_n + 1)},$$

$$\sigma_{ij}^h = \sum_{n=1}^{\infty}\sum_{k=1}^{3}[\beta_n b_{kn}\tau_{ijk}z_k^{\delta_n} + \beta_n b_{(k+3)n}\overline{\tau}_{ijk}\overline{z}_k^{\delta_n}], \tag{2.6}$$

$$\tau_{ijk} = \sum_{m=1}^{3}(C_{ijm1} + \mu_k C_{ijm2})\nu_{mk} \quad \text{(no sum on } k\text{)},$$

where β_n are real or complex free constants to be determined from the far field conditions; ν_{ik} is an eigenvector corresponding to an eigenvalue μ_k for the following eigenvalue equation resulting from equilibrium equation (2.5)

$$[C_{i1j1} + \mu_k(C_{i1j2} + C_{i2j1}) + \mu_k^2 C_{i2j2}]\nu_{jk} = 0.$$

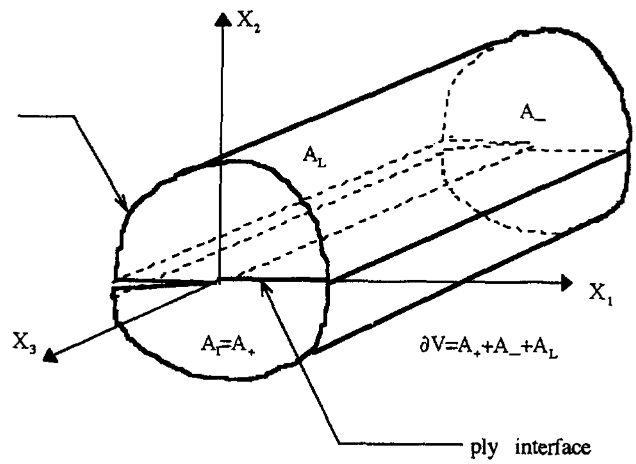

Figure 2. An arbitrary cylindrical domain V_A containing a delamination crack in a laminated composite strip under generalized plane deformation.

Moreover b_{kn} is an eigenvector for an eigenvalue δ_n of an eigenvalue equation, which is determined from the following near-field conditions

$$\sigma_{2j} - \sigma'_{2j} = 0 \text{ and } u_j - u'_j = 0 \quad \text{on } x_2 = 0 \text{ and } x_1 > 0 \quad \text{(along the ply interface)},$$

$$\sigma_{2j} = 0 \quad \text{on } x_2 = 0^+ \text{ and } x_1 < 0 \quad \text{(along the upper crack face)},$$

$$\sigma'_{2j} = 0 \quad \text{on } x_2 = 0^- \text{ and } x_1 < 0 \quad \text{(along the lower crack face)},$$

where the prime '′' indicates the quantity for the lower ply.

3. J-integral and mutual interaction integral for delamination cracks

For an edge delamination crack under generalized plane deformation described in Section 2, we can derive the expression for the J-integral, which is equivalent to the crack growth driving force or the energy release rate. Consider a three-dimensional cylindrical domain V_A bounded by two opposite cross-sectional areas A_+ and A_- containing the crack-tip and by the lateral surface A_L of unit length, as shown in Figure 2. We begin with the expression for the J-integral for the three-dimensional domain V_A

$$J = \int_{\partial V_A} \left(W n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) \mathrm{d}A$$

where t_i is a traction vector component, and $\partial V_A = A_+ + A_- + A_L$ indicates the entire boundary of the domain V_A. Note that the size or shape of the surface integral domain does not matter in the above expression because the integral is surface independent in the absence of body force. Substituting the expressions (2.1) for the displacement components, and noting that the state of stress and strain is invariant along the x_3 axis, we obtain the following result

$$J = \int_{C_\mathrm{I}} \left(W \delta_{1j} - \sigma_{ij} \frac{\partial U_i}{\partial x_1} \right) n_j \, \mathrm{d}s - \int_{A_\mathrm{I}} (A_4 \sigma_{23} + A_2 \sigma_{33}) \, \mathrm{d}A \qquad (3.1)$$

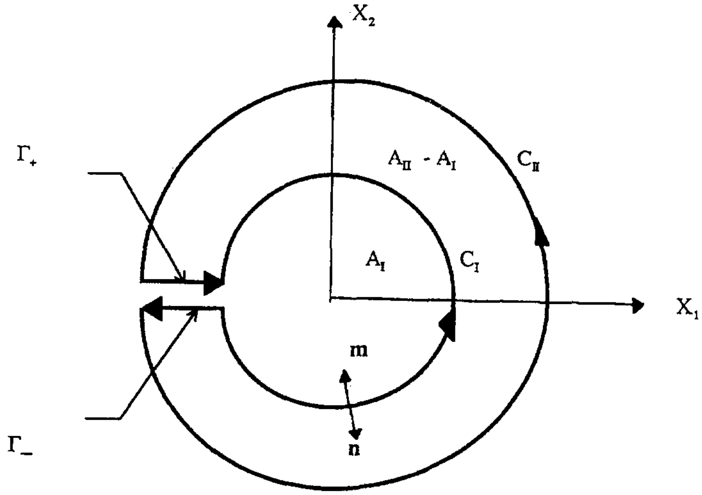

Figure 3. Integration domains for the domain integral.

where $A_{\rm I}$ indicates A_+, that is, an arbitrary part of the cross-section containing the crack tip, and $C_{\rm I}$ is the boundary contour of $A_{\rm I}$. In terms of the cross-sectional distortion $U_i(x_1, x_2)$ there appears the area integral term, in addition to the contour integral term, in the expression for the J-integral of generalized plane deformations wherein A_2 or A_4 is nonzero. In fact the term inside the parenthesis in the first contour integral is not divergence free, but its divergence yields the integrand of the second (area) integral term. As a whole, this makes the above expression independent of domain $A_{\rm I}$. For more accurate numerical computation, however, the following domain integral representation [17] is more appropriate (see Appendix B for detail):

$$J = -\int_{A_{\rm II}-A_{\rm I}} (\sigma_{33}A_2 + \sigma_{23}A_4) q \, \mathrm{d}A \\ - \int_{A_{\rm II}-A_{\rm I}} (W\delta_{1j} - \sigma_{ij}U_{i,1}) q_{,j} \, \mathrm{d}A - \int_{A_{\rm I}} (\sigma_{33}A_2 + \sigma_{23}A_4) \, \mathrm{d}A, \quad (3.2)$$

where δ_{ij} is the Kronecker delta; $A_{\rm II}$ is the area surrounded by the path $C_{\rm II}$, as shown in Figure 3; moreover, q is a continuously differentiable weight function which is equal to 1 on $C_{\rm I}$ and to 0 on $C_{\rm II}$. We could make the inner domain shrink to an infinitesimal one, but we prefer to leave it as a finite domain for convenience of numerical computation as in Figure 3.

For interfacial cracks such as delamination cracks, the interfacial resistance to crack growth is mode dependent in general: the growth resistance or the critical energy release rate is known to increase as the mode II becomes dominant over mode I [14]. Hence the J-integral or the energy release rate itself is not enough to judge how critical the state of the crack-tip stress field is. We need to introduce the parameters characterizing the near tip traction field in more detail. The state of three asymptotic traction components is represented by Suo's intensities [4]. Following this, for an opened interfacial crack Kim and Im [7] wrote the asymptotic traction as

$$\mathbf{t}^s(r, 0) = \frac{1}{\sqrt{2\pi r}} \{ K r^{i\eta} \mathbf{w} + \overline{K} r^{-i\eta} \overline{\mathbf{w}} + K_3 \mathbf{w}^0 \}, \quad (3.3)$$

as r approaches zero, where

$$K = \sqrt{2\pi}\beta_1, \qquad K_3 = 2\sqrt{2\pi}\beta_3, \qquad \eta = \mathrm{Im}(\delta_1),$$
$$w_j = \sum_{k=1}^{3}(b_{k1}\tau_{2jk} + b_{(k+3)1}\overline{\tau}_{2jk}), \qquad w_j^0 = \mathrm{Re}\left(\sum_{k=1}^{3} b_{k3}\tau_{2jk}\right), \tag{3.4}$$

where $\delta_{1,2} = -\frac{1}{2} \pm i\eta$ and $\delta_3 = -\frac{1}{2}$ are the crack-tip singularities with the imaginary part η and 0, respectively [7]. Here $\mathbf{t}^s$ is the singular traction vector, and $\mathbf{w}$ and $\mathbf{w}^0$ are the eigenvectors from an interfacial crack problem [4] (see Appendix A). For delamination cracks in composite laminates under generalized plane deformation, the above stress intensity factors, K and K_3, that is, three real intensities $\mathrm{Re}[K]$, $\mathrm{Im}[K]$ and K_3, fully characterize the near tip traction field for an interfacial delamination crack. We need the three scaling parameters $\mathrm{Re}[K]$, $\mathrm{Im}[K]$ and K_3 for representing the asymptotic traction state when the imaginary part of singularity η is given. However, the complex intensity factor K has the length scale dependency [11], and we therefore use the following stress intensity factors K_1 and K_2, as suggested by Rice [11], based upon a specific reference length $\hat{r}$

$$\mathbf{t}^s(r,0) = \frac{1}{\sqrt{2\pi r}}\{(K_1 + iK_2)(r/\hat{r})^{i\eta}\mathbf{w} + (K_1 - iK_2)(r/\hat{r})^{-i\eta}\overline{\mathbf{w}} + K_3\mathbf{w}^0\}, \tag{3.5}$$

where the definitions of K_1 and K_2 are apparent from (3.3) and (3.5). As seen from (3.5), the near field traction for an opened delamination crack is not related to the three stress intensity factors in the classical sense; that is, the stress intensities are not decomposed into the three independent modes of the classical fracture mechanics, but the three modes are inherently coupled with one another. Moreover, the eigenvectors $\mathbf{w}$ and $\mathbf{w}^0$ are dependent upon the material properties as well as upon how the coefficient vector b_{kn} is normalized.

The stress intensity factor for an interfacial crack, particularly for an opened interfacial crack, is not easy to obtain from the displacement based regular FEM because of the oscillatory singularity, as opposed to the stress intensity factors in the classical sense for cracks in a homogeneous material for which a typical singular element approach or a regular FEM with the aid of extrapolation turns out to be very successful in calculating the stress intensity factors. Hence, for interfacial cracks in laminated composite strips some special techniques have been used for accurate calculation of the stress intensity factor, for example, boundary collocation method (Wang [1]), or singular hybrid FEM (Wang and Yuan [6]). Recently, Suo's complex stress intensities for delamination cracks and the so called free edge stress intensities for free edge in a laminated composite strip under generalized plane deformations have been calculated via hybrid FEM [7] and via enriched FEM [8]. The aforementioned special numerical methods involve considerable complexities in implementation as well as in formulation although they retain good accuracy. Because of the oscillatory singularity, a simple singular element that possesses the inverse square root singularity is not applicable for an opened delamination crack. On the other hand, the singular hybrid FEM as well as the enriched FEM is rather complicated in that the asymptotic solution in the form of eigenfunction series has to be incorporated into the elements. In the paragraph to follow, we describe a procedure for calculating the stress intensities for an opened edge delamination crack under generalized plane deformation by way of the J-integral derived in (3.1) and the so called mutual interaction integral [10], which could be related to earlier works [15, 16].

The energy release rate introduced in [4] is

$$G = \alpha(K_1^2 + K_2^2) + \beta K_3^2, \tag{3.6}$$

where

$$\alpha = \frac{\mathbf{w}^T \cdot (\mathbf{R} + \overline{\mathbf{R}})\mathbf{w}}{4\cosh^2 \pi\eta}, \qquad \beta \frac{1}{8}\mathbf{w}^{0T} \cdot (\mathbf{R} + \overline{\mathbf{R}})\mathbf{w}^0$$

and $\mathbf{R}$ is a Hermitian matrix (see Appendix A).

Consider two independent delamination crack problems: let the superscript A indicate the generalized plane deformation field under consideration and the superscript B an arbitrary auxiliary field. For an auxiliary field, it is most convenient to take a simple generalized plane strain field, for which the deformation parameters A_i in (2.1) identically vanish. We can suppose another equilibrium state obtained by superposing the two equilibrium states A and B, and let the superscript C indicate this superposed equilibrium state. The J-integral for the state C, which is equivalent to (3.6) of the energy release rate G, can be written as

$$\begin{aligned} J^C &= \alpha[(K_1^C)^2 + (K_2^C)^2] + \beta(K_3^C)^2 \\ &= J^A + J^B + M^{(A,B)}, \end{aligned} \tag{3.7}$$

where

$$M^{(A,B)} = 2[\alpha(K_1^A K_1^B + K_2^A K_2^B) + \beta K_3^A K_3^B]$$

is the mutual interaction integral and is derived in detail in Appendix B. In the domain integral representation it may be written as

$$\begin{aligned} M^{(A,B)} &= \int_{A_{\mathrm{II}} - A_{\mathrm{I}}} [-\sigma_{ij}^A \varepsilon_{ij}^B q_{,1} + (\sigma_{jk}^A U_{j,1}^B + \sigma_{jk}^B U_{j,1}^A) q_{,k}]\, \mathrm{d}A \\ &\quad - \int_{A_{\mathrm{II}} - A_{\mathrm{I}}} (\sigma_{23}^B A_4 + \sigma_{33}^B A_2) q\, \mathrm{d}A - \int_{A_{\mathrm{I}}} (\sigma_{23}^B A_4 + \sigma_{33}^B A_2)\, \mathrm{d}A, \end{aligned} \tag{3.8}$$

where A_{I}, A_{II} are the areas surrounded by each of the paths C_{I}, C_{II} in Figure 3 and q is a weight function as in (3.2). In the derivation above, it is crucial to employ the same normalization in calculating the eigenvectors $\mathbf{w}$ and $\mathbf{w}^0$ (or b_{kn}) of (3.4) for the elastic states A and B. Now let the superscript A indicate a stress field under consideration for which the solution including stress intensities is not yet known, and B_1, B_2, B_3 the solution for each of any three independent stress fields which are already known. From (3.7), the mutual integrals for the state A and each of the states B_1, B_2, B_3 are written as

$$M^{(A,B_j)} = 2[\alpha(K_1^A K_1^{B_j} + K_2^A K_2^{B_j}) + \beta K_3^A K_3^{B_j}] \quad (j = 1, 2, 3). \tag{3.9}$$

The left-hand side of these equations can be calculated, employing the expression (3.8), with the aid of a displacement based regular finite element analysis. For simplicity we employ the asymptotic solution field (2.6), which is a generalized plane strain field, retaining only singular terms in order to generate three auxiliary stress fields B_1, B_2, and B_3. That is, we prescribe three sets of stress intensities $K_1^{B_j}$, $K_2^{B_j}$ and $K_3^{B_j} (j = 1, 2, 3)$ in (2.6), and we then

Table 1. The eigenvalues μ_k and stress singularities δ_s

Laminates	[45/−45]		[60/−45]	
	45	−45	60	−45
μ_1	$i\,0.83467$	$i\,0.83467$	$i\,0.75525$	$i\,0.83467$
μ_2	$i\,3.4719$	$i\,3.4719$	$i\,4.1930$	$i\,3.4719$
μ_3	$i\,1.1644$	$i\,1.1644$	$i\,1.2206$	$i\,1.1644$
δ_1	$-\frac{1}{2}+i\,1.049265245\times 10^{-2}$		$-\frac{1}{2}+i\,1.354390843\times 10^{-2}$	
δ_2	$-\frac{1}{2}-i\,1.049265245\times 10^{-2}$		$-\frac{1}{2}-i\,1.354390843\times 10^{-2}$	
δ_3	$-\frac{1}{2}$		$-\frac{1}{2}$	

obtain the corresponding stress fields, which are the auxiliary elastic fields (see Appendix C). Then Eqns. (3.9) lead to a set of three linear equations in the three unknowns $K_i^A (i = 1, 2, 3)$ when its left hand side is computed from (3.8).

4. Numerical examples and discussion

In this section, we take an edge delamination crack in a laminated composite strip under generalized plane deformation, and calculate the energy release rate via the J-integral for different types of loadings. Moreover, we employ the mutual interaction integral to compute the aforementioned stress intensities and the mode mixity. From these fracture mechanics parameters, we will examine the stability of crack growth and the influence of geometry upon fracture behavior.

For numerical examples, we choose [45/−45] and [60/−45] laminated with an edge delamination crack on their left edge under various loadings within the range of generalized plane deformation. The types of loadings considered and the cross-sectional geometry for the laminates are shown in Figure 4. Moreover, we employ the following material data for the graphite epoxy T300/5028 [5],

$$E_L = 134\,\text{GPa}, \qquad E_T = E_Z = 10.2\,\text{GPa},$$
$$G_{LT} = G_{LZ} = 5.52\,\text{GPa}, \qquad G_{TZ} = 3.43\,\text{GPa},$$
$$\nu_{LT} = \nu_{LZ} = 0.3, \qquad \nu_{TZ} = 0.49,$$

where L, T and Z indicate the principal material axes along fiber, transverse and thickness direction, respectively. Table 1 shows the eigenvalues μ_k for each of the ply orientations for this material and the eigenvalues δ_s associated with delamination singularities for the two laminates.

To confirm the present scheme, we compute the J-integral values and the stress intensity factors from a regular FEM in conjunction with the mutual interaction integral technique, and compare these results with the energy release rates and the stress intensities obtained from the enriched FEM [8]. Displacement based regular finite element formulation for the present problem can be carried out in a straightforward manner: the cross-sectional distortion $U_i(x_1, x_2)$ is taken for the nodal variables, and the remaining terms, related to deformation parameters A_i, contribute to the resulting finite element equation just like initial strains [7, 18]. The mesh configuration employed in the present finite element analysis is schematically

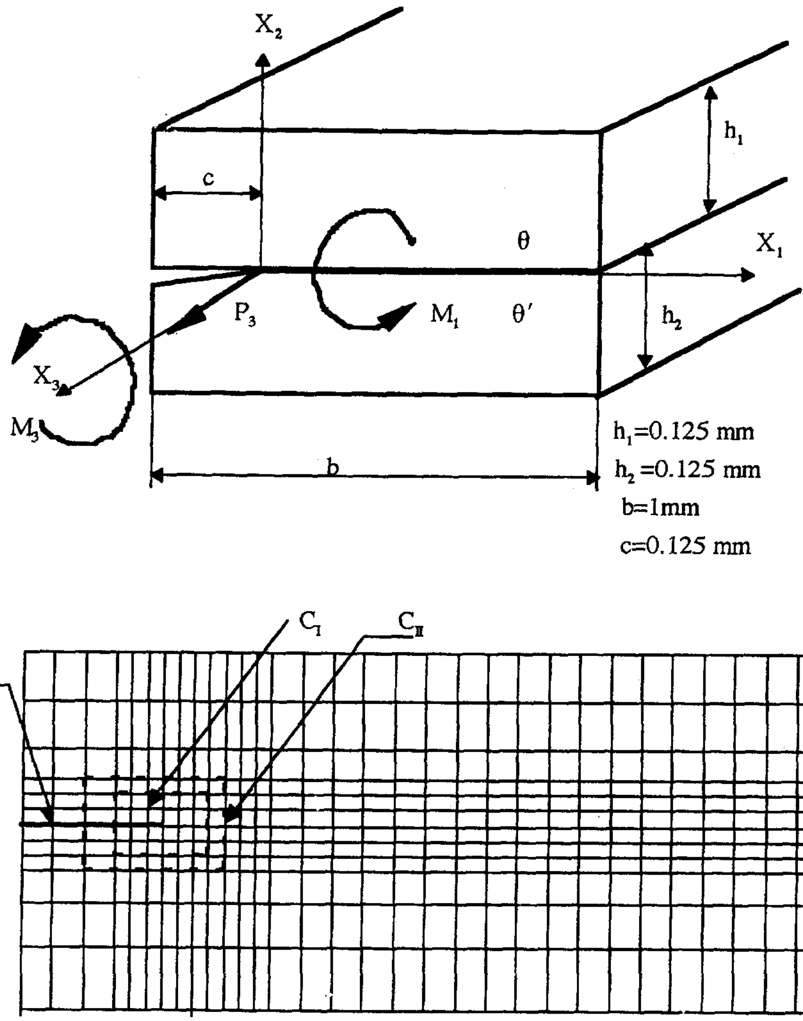

Figure 4. The laminated composite strip of $[\theta/\theta']$ and a schematic description of the finite element mesh for its cross section ($[\theta/\theta']$ = [45/-45] or [60/-45]).

depicted in Figure 4. The entire cross-section was discretized into the total 396 eight-noded isoparametric elements. It has been checked that this discretization is sufficient for convergence of FEM solution. Moreover, the J-integral and the mutual integral $M^{(A,B)}$ turn out to be independent of the choice of the domains A_{I} and A_{II} wherein A_{II} is apart from A_{I} by one element spacing, and the domains chosen for computation are shown in Figure 4. We have chosen three loadings, including uniaxial compression ($P_3 = -1$), bending about the x_1-axis ($M_1 = 1$ or -1) and torsion ($M_3 = -1$). Under each of these loadings, the crack faces are found to be opened, so that the present opened crack model is applicable. To obtain the solution in terms of loading, not in terms of deformation parameters A_i, Newton type iteration is required because of the possibility of the crack closure; however, the resultant axial force and moments on the cross section are linearly related to the deformation parameters A_i in the absence of the crack closure [7].

Table 2 shows the comparison of the J-integral values obtained from the present regular FEM and the energy release rate obtained from the enriched FEM [8]. The energy release

Table 2. Comparison of the energy release rate and the J-integral for the [45/−45] and [60/−45] laminated composite strips

Laminates	Loadings R	$J/\|R\|^2$ (present scheme)	$G/\|R\|^2$ (enriched FEM [8])
	$R = P_3 = -1$	0.1535	0.1552
[45/ − 45]	$R = M_1 = 1$	1.428	1.464
	$R = M_3 = -1$	10.02	10.10
	$R = P_3 = -1$	0.0914	0.0924
[60/ − 45]	$R = M_1 = -1$	1.101	1.130
	$R = M_3 = -1$	12.17	12.26

Table 3. Comparison of the stress intensity factors from the present scheme with those from the enriched FEM. (K_1 and K_2 are defined based upon $\hat{r} = c/50$)

Laminates	Loading	Present scheme			Enriched FEM [8]		
	R	K_1/R	K_2/R	K_3/R	K_1/R	K_2/R	K_3/R
	$R = P_3 = -1$	−0.272	1.353	0.0	−0.2764	1.369	0.0
[45/ − 45]	$R = M_1 = 1$	0.0	0.0	−4.749	0.0	0.0	−4.826
	$R = M_3 = -1$	−7.616	−8.139	0.0	−7.764	−8.168	0.0
	$R = P_3 = -1$	−0.1879	0.9645	0.4578	−0.1909	0.9759	0.4621
[60/ − 45]	$R = M_1 = -1$	−0.8831	3.093	−1.935	−0.9053	3.144	−1.970
	$R = M_3 = -1$	−8.338	−7.301	−5.745	−8.490	−7.325	−5.772

rate from the enriched FEM, wherein stress intensities are directly computed from the FEM solution, has been computed via (3.6). The solutions for two different approaches show an excellent agreement. Table 3 shows the stress intensities obtained from the two different approaches. The stress intensity factors K_1 and K_2 are defined based upon the reference length $\hat{r} = c/50$, where c is the crack length. Two solutions are in excellent agreement again.

In the enriched FEM, the general description of which is given in [13], the strain matrix terms or stiffness matrix terms involving the crack-tip singular solution requires very accurate numerical integration as well as complex formulation. For this, very high order Gaussian rule is employed and moreover the element aspect ratio has to be maintained close to 1 [8]. This is the case also for the singular hybrid FEM [6, 7]. However, the present scheme, wherein a regular FEM and the mutual integral technique is used, does not involve such complexities and its implementation is simple and straightforward.

For $[\theta/\theta']$ composite laminates under generalized plane deformation, three real scaling parameters K_1, K_2 and K_3 fully characterize the near tip traction field for an interfacial delamination crack when the reference length $\hat{r}$ is fixed. The relative values of the three traction components along the interface or the mode mixity plays an important role in the fracture initiation or in the onset of crack growth because the crack growth resistance is mode dependent. The mode mixity will in general depend upon the three scaling parameters K_1, K_2 and K_3 and moreover, it varies along the interface depending upon a ligament distance from the crack tip for a given set of K_1, K_2 and K_3. One convenient choice of defining the mode mixity may be given in terms of the traction components at the ligament distance of reference length $\hat{r}$ from the crack tip. Following this, we may define the mode mixity in terms of the phase angles Ψ_1 and Ψ_2

$$\Psi_1 = \tan^{-1}\left(\frac{k_2}{k_1}\right), \qquad \Psi_2 = \cos^{-1}\left(\frac{k_3}{\sqrt{k_1^2 + k_2^2 + k_3^2}}\right), \tag{4.1}$$

where

$$k_i = (K_1 + iK_2)w_i + (K_1 - iK_2)\overline{w}_i + K_3 w_1^0.$$

Now the three real parameters G, Ψ_1 and Ψ_2 can replace the three scalar stress intensity factors K_1, K_2 and K_3, and the crack growth criterion may be stated as

$$G = G_c(\Psi_1, \Psi_2), \tag{4.2}$$

where G_c is the critical energy release rate, which is a material constant depending upon the mode mixity.

Figures 5 to 10 show the variation of the energy release rate and the mode mixity versus the crack length in the [45/−45] and [60/−45] composite laminates subjected to the aforementioned three types of loadings. Under each of these loadings the crack faces turn out to be opened, so that our opened crack model may be applicable. Here the mode mixity in terms of the phase angles is defined based upon a fixed reference length $\hat{r}/b = 0.01$ regardless of the crack lengths. For the uniaxial compression, the energy release rate undergoes a sharp increase at an early stage of crack growth and then reaches the peak value before decreasing monotonically in both the [45/−45] and [60/−45] laminates. If we neglect the influence of the mode mixity change, which is not prominent near the peak value of energy release rate, there may exist a crack arrest mechanism such that the initial unstable crack growth with increasing driving force is arrested due to the decreasing driving force subsequent to the peak value. For bending and torsion, however, the energy release rate monotonically increase as the crack grows in both the [45/−45] and [60/−45] laminates, and therefore the crack may undergo unstable growth, so that the crack growth behavior for these two loadings may be different from the case of the uniaxial compression. Strictly speaking, the exact growth behavior can be understood only when it is known how the resistance to crack growth depends upon the mode mixity because the mode mixity is, even if not significantly, changing as the cracks grow.

Appendix A

The eigenvectors $\mathbf{w}$ and $\mathbf{w}^0$ are determined from the following eigenvalue problems obtained from the near field conditions - the traction free conditions on the crack faces and the continuity conditions along the ply interface.

$$\overline{\mathbf{R}}\mathbf{w} = e^{i\pi\eta}\mathbf{R}\mathbf{w}, \qquad \overline{\mathbf{R}}\mathbf{w}^0 = \mathbf{R}\mathbf{w}^0, \tag{A.1}$$

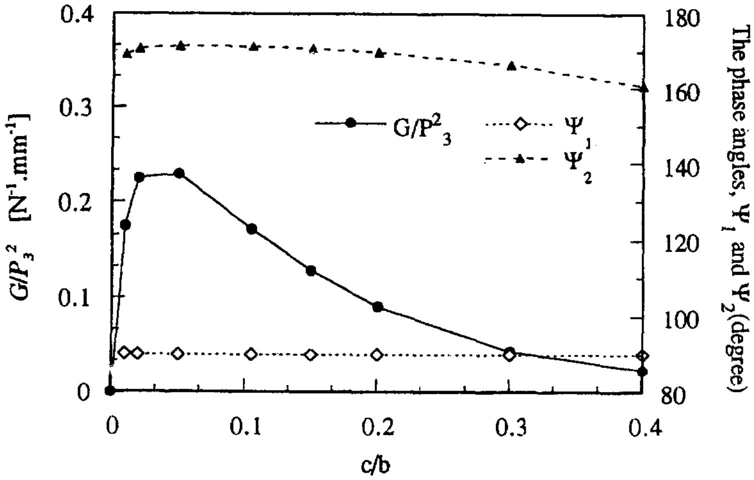

Figure 5. The energy release rate and the mode mixity in terms of the phase angles versus the crack length for compression along the strip axis for the [45/−45] laminate.

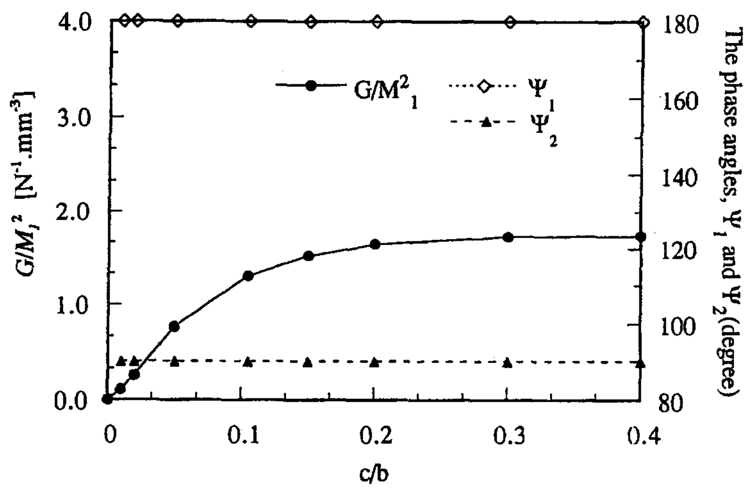

Figure 6. The energy release rate and the mode mixity in terms of the phase angles versus the crack length for bending about the x_1-axis for the [45/−45] laminate.

where

$$\mathbf{R} = \mathbf{B} + \overline{\mathbf{B}}', \qquad B_{rs=} = i \sum_{k=1}^{3} v_{rk} L_{ks}^{-1},$$

$$L_{rs} = \tau_{r2s} = \sum_{k=1}^{3} (C_{r2k1} + \mu_s C_{r2k2}) v_{ks} \quad (\text{no sum on } s, \text{ and } r, s = 1, 2, 3),$$

and $\mathbf{w}$ and $\mathbf{w}^0$ are related to b_{kn} as in (3.3). Moreover, η is the imaginary part of the singular eigenvalues δ_1 and δ_2; μ_s and v_{ks} are referred to Section 2, and the prime '′' indicates the quantity for the lower ply. Kim and Im [7] proposed a normalization of $\mathbf{w}$ and $\mathbf{w}^0$ for $[\theta/-\theta]$ composite laminates as follows

$$\mathbf{w} = [^*, -\tfrac{1}{2}, ^*]^T, \qquad \mathbf{w}^0 = [1, ^*, ^*]^T,$$

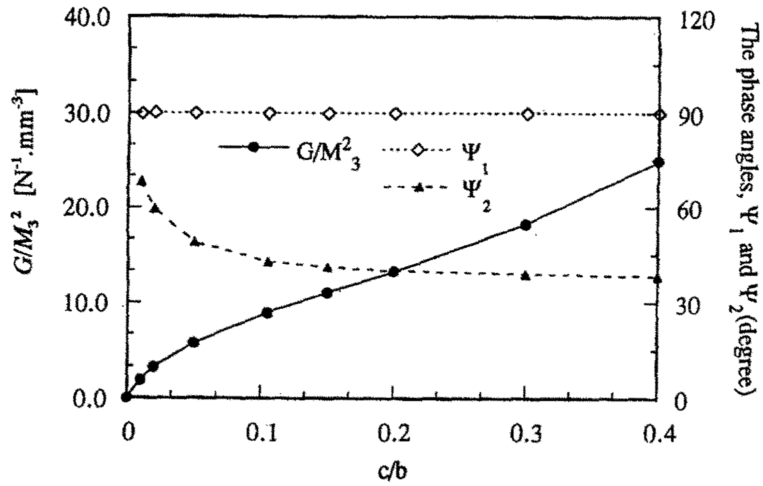

Figure 7. The energy relase rate and the mode mixity in terms of the phase angles versus the crack length for torsion about the strip axis for the [45/−45] laminate.

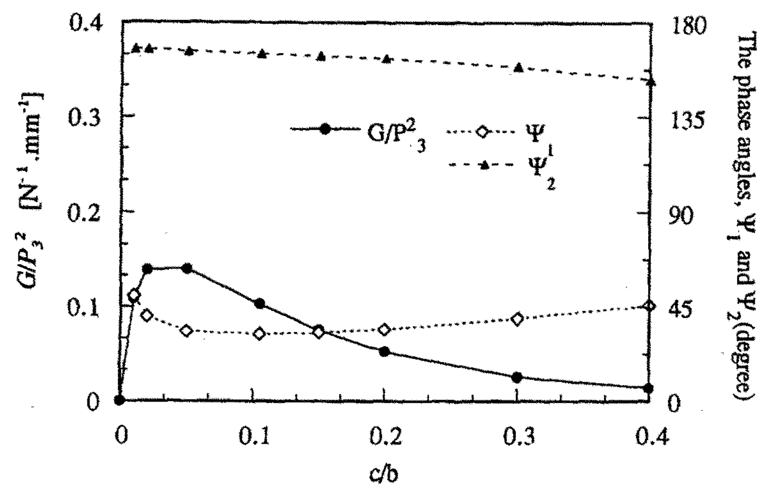

Figure 8. The energy release rate and the mode mixity in terms of the phase angles versus the crack length for compression along the strip axis for the [60/−45] laminate.

where (*) signifies numbers determined by (A.1) of the eigenvalue problem. From (3.3), it is apparent that this choice can be made by choosing appropriate normalization for the eigenvector b_{kn} in (2.6).

Appendix B

Consider a domain as shown in Figure 3. The J-integral given by (3.1) may be written as

$$J = -\int_{C_{\rm II}+\Gamma^+-C_{\rm I}+\Gamma^-}\left(W\delta_{1j}-\sigma_{ij}\frac{\partial U_i}{\partial x_1}\right)qm_j\,\mathrm{d}s-\int_{A_{\rm I}}(A_4\sigma_{23}+A_2\sigma_{33})\,\mathrm{d}A,$$

where q is a continuously differentiable weight function which is equal to 1 on $C_{\rm I}$ and to 0 on $C_{\rm II}$. Note that the integrand of the first integral in the above expression vanishes identically on the crack faces Γ^+ and Γ^-, and that the unit vector $\mathbf{m}$ is an outward normal vector to the

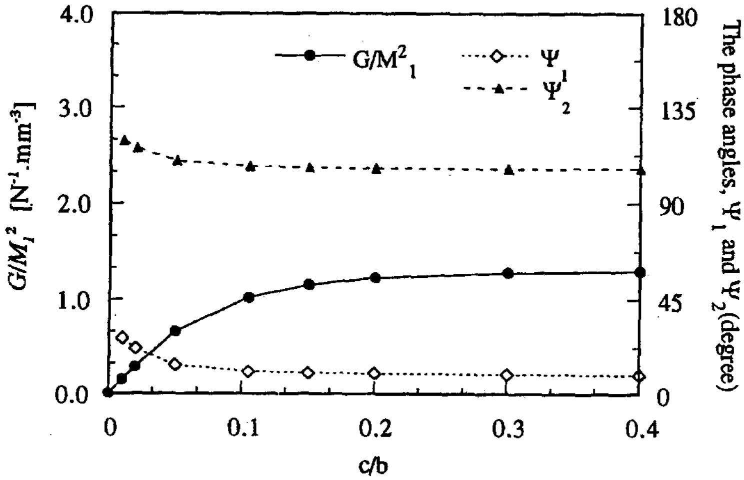

Figure 9. The energy release rate and the mode mixity in terms of the phase angles versus the crack length for bending about the x_1-axis for the [60/−45] laminate.

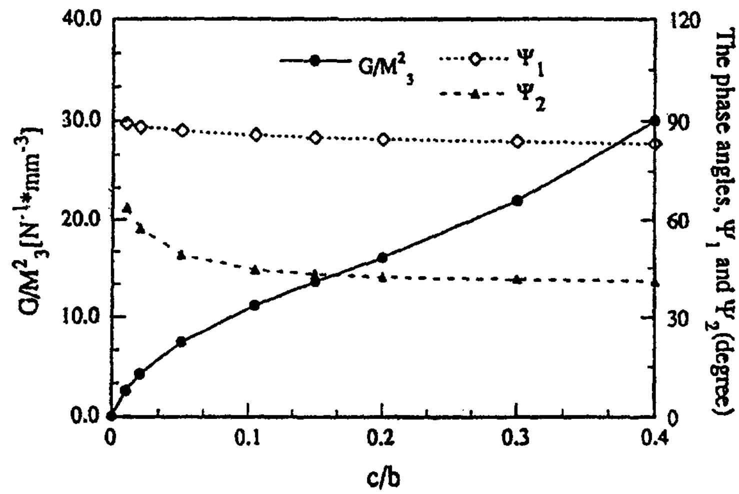

Figure 10. The energy release rate and the mode mixity in terms of the phase angles versus the crack length for torsion about the strip axis for the [60/−45] laminate.

boundary of $A_{\rm II} - A_{\rm I}$, and its direction is opposite to $\mathbf{n}$ on $C_{\rm I}$, as shown in Figure 3. Applying the divergence theorem, we have

$$
\begin{aligned}
J = &-\int_{A_{\rm II}-A_{\rm I}} (W\delta_{1j} - \sigma_{ij}U_{i,1})q_{,j}\,\mathrm{d}A \\
&-\int_{A_{\rm II}-A_{\rm I}} (W_{,j}\delta_{1j} - \sigma_{ij,j}U_{i,1} - \sigma_{ij}U_{i,1j})q\,\mathrm{d}A \\
&-\int_{A_{\rm I}} (A_4\sigma_{23} + A_2\sigma_{33})\,\mathrm{d}A.
\end{aligned}
$$

Noting that $\sigma_{ij,j} = 0, W_{,j} = \sigma_{km}\varepsilon_{km,j} = \sigma_{km}u_{k,mj}$, and employing (2.1), we can obtain the following form

$$
\begin{aligned}
J = &-\int_{A_{\mathrm{II}}-A_{\mathrm{I}}} (\sigma_{33}A_2 + \sigma_{23}A_4)q\,\mathrm{d}A \\
&-\int_{A_{\mathrm{II}}-A_{\mathrm{I}}} (W\delta_{1j} - \sigma_{ij}U_{i,1})q_{,j}\,\mathrm{d}A \\
&-\int_{A_{\mathrm{I}}} (\sigma_{33}A_2 + \sigma_{23}A_4)\,\mathrm{d}A.
\end{aligned}
$$

We take superposition $\sigma_{ij}^C = \sigma_{ij}^A + \sigma_{ij}^B, U_i^C = U_i^A + U_i^B$, and plug this into the above domain integral form. We then have

$$
\begin{aligned}
J = &-\int_{A_{\mathrm{II}}-A_{\mathrm{I}}} [(\sigma_{33}^A + \sigma_{33}^B)A_2 + (\sigma_{23}^A + \sigma_{23}^B)A_4]q\,\mathrm{d}A \\
&-\int_{A_{\mathrm{II}}-A_{\mathrm{I}}} [\tfrac{1}{2}(\sigma_{km}^A + \sigma_{km}^B)(\varepsilon_{km}^A + \varepsilon_{km}^B)q_{,j} - (\sigma_{ij}^A + \sigma_{ij}^B)(U_{i,1}^A + U_{i,1}^B)q_{,j}]\,\mathrm{d}A \\
&-\int_{A_{\mathrm{I}}} [(\sigma_{33}^A + \sigma_{33}^B)A_2 + (\sigma_{23}^A + \sigma_{23}^B)A_4]\,\mathrm{d}A.
\end{aligned}
$$

Sorting out the terms forming J^A and J^C, we may write J^C as

$$J^C = J^A + J^B + M^{(A,B)},$$

where

$$
\begin{aligned}
M^{(A,B)} = &\int_{A_{\mathrm{II}}-A_{\mathrm{I}}} [-\sigma_{ij}^A\varepsilon_{ij}^B q_{,1} + (\sigma_{jk}^A U_{j,1}^B + \sigma_{jk}^B U_{j,1}^A)q_{,k}]\,\mathrm{d}A \\
&-\int_{A_{\mathrm{II}}-A_{\mathrm{I}}} (\sigma_{23}^B A_4 + \sigma_{33}^B A_2)q\,\mathrm{d}A - \int_{A_{\mathrm{I}}} (\sigma_{23}^B A_4 + \sigma_{33}^B A_2)\,\mathrm{d}A.
\end{aligned}
$$

Appendix C

The K-field, which is used for the auxiliary field in Section 3, is given as

$$
\begin{aligned}
u_i^{B_j} = &\frac{4}{\sqrt{2\pi}}\sum_{k=1}^{3} \mathrm{Re}\left[\frac{1}{1+i\eta}(K_1^{B_j} + iK_2^{B_j})\hat{r}^{-i\eta}(b_{k1}v_{ik}z_k^{(1/2)+i\eta}\right. \\
&\left. + b_{(k+3)1}\overline{v}_{ik}\overline{z}_k^{(1/2)+i\eta})\right] + \frac{2}{\sqrt{2\pi}}K_3^{B_j}\sum_{k=1}^{3} \mathrm{Re}(b_{k3}v_{ik}z_k^{(1/2)}),
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{im}^{B_j} = &\frac{2}{\sqrt{2\pi}}\sum_{k=1}^{3} \mathrm{Re}[(K_1^{B_j} + iK_2^{B_j})\hat{r}^{-i\eta}(b_{k1}\tau_{imk}z_k^{-(1/2)+i\eta} \\
&+ b_{(k+3)1}\overline{\tau}_{imk}\overline{z}_k^{-(1/2)+i\eta})] + \frac{K_3^{B_j}}{\sqrt{2\pi}}\sum_{k=1}^{3} \mathrm{Re}(b_{k3}\tau_{imk}z_k^{-(1/2)}).
\end{aligned}
$$

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