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Enriched finite element analysis for a delamination crack in a laminated composite strip

I. Jeon, Y. Kim, S. Im

Abstract Edge delamination cracks in laminated composite strips are analyzed with the aid of the enriched finite element method, wherein the asymptotic singular solution for a delamination crack is incorporated into finite elements. The strip is assumed to be in the state of generalized plane deformations including extension (compression), bending or torsion. Comparison of the numerical results with those from other methods is made to confirm the solution. The crack growth stability is examined for a couple of ply orientations in terms of the energy release rate and mode mixity.

1 Introduction

Geometric or material discontinuities in a composite laminate such as, free edge, transverse crack or delamination, induce singular stress fields, and oftentimes such discontinuity points of singular stress field lead to more extensive damage as static or fatigue loading increases. Therefore there have been intensive efforts for providing an insight into the region near these discontinuities from the viewpoint of elastic fracture mechanics, and many numerical schemes for calculating the stress intensities for singular stress fields have been employed.

Among others, several refined numerical techniques in conjunction with asymptotic singular solution have been employed to calculate the stress intensity for free edge or delamination problems in a laminate composite; citing a few, but not limited to the boundary collocation method (Wang and Choi (1983) or Wang (1984)), hybrid FEM (Wang and Yuan (1983)), and enriched FEM (Stolarski and Chiang (1989)). On the other hand, Chow and Atluri (1995a) have reported the calculation of the stress intensity factors using virtual crack closure integral, wherein the crack opening displacement and the nodal forces at and ahead of the crack tip are employed for computing the stress intensities. Moreover, recently there have been reported the calculation of Suo's complex stress intensities (1990) for delamination cracks in composite laminates via the hybrid FEM or via the two-state conservation law; for examples, Kim and Im (1995), Chow, Beom and Atluri (1995), Jeon, Cha and Im (1995), and Chow and Atluri (1995b).

In this paper we are concerned with a delamination crack in a laminated composite strip by way of the enriched FEM. The enriched FEM was developed by Mote (1971), and successfully used for analysis of singular stress fields in composite laminates by Benzly (1974) and Chen (1985). The extensive description of the scheme may be found in Atluri and Nakagaki (1986). One of the recent applications of this scheme is the free edge of composite laminates by Stolaski and Chiang (1989).

We first summarize the governing equation and the asymptotic singular solution for deformation of a laminated composite strip with an edge delamination crack. Employing the asymptotic solution for an opened crack, and following the standard formulation for the scheme as in Stolarski and Chiang (1989) or in Atluri and Nakagaki (1986), we obtain the finite element equation. The relationship between the stress intensity and the energy release rate is discussed for an opened crack. Moreover, an appropriate definition for the mode mixity is introduced. The convergence of FEM solution is checked versus the mesh discretizations, and the solution is confirmed by way of comparison to the results from other numerical methods such as the method of the two-state J-integral (Jeon, Cha and Im (1995)).

For a couple of ply orientations and various loadings, the energy release rate and the mode mixity are computed for varying crack length, and the crack growth stability is discussed in terms of these parameters. Finally some remarks are made regarding the delamination crack behavior under various loadings.

2 The statement of the problem and the basic equations

Consider a sufficiently long laminated composite strip under generalized plane deformation wherein the state of stress and strain does not vary along the axial coordinate of strip. The generalized plane deformation of the strip may then occur when it is subjected to the end loading of axial force, pure bending moment and/or torsion with no shear forces upon the lateral surface of strip (see Fig. 1). Included in the generalized plane deformation is also the generalized plane strain problems, wherein self equilibrated lateral traction orthogonal to the generator of strip is applied uniformly independent of the coordinate along the strip length. We suppose an edge delamination crack in a laminated composite strip under generalized plane deformation, as shown in Fig. 1. We take coordinate system with origin at a crack tip such that the x_3-axis is along the axial direction of strip and the x_2-axis is along the

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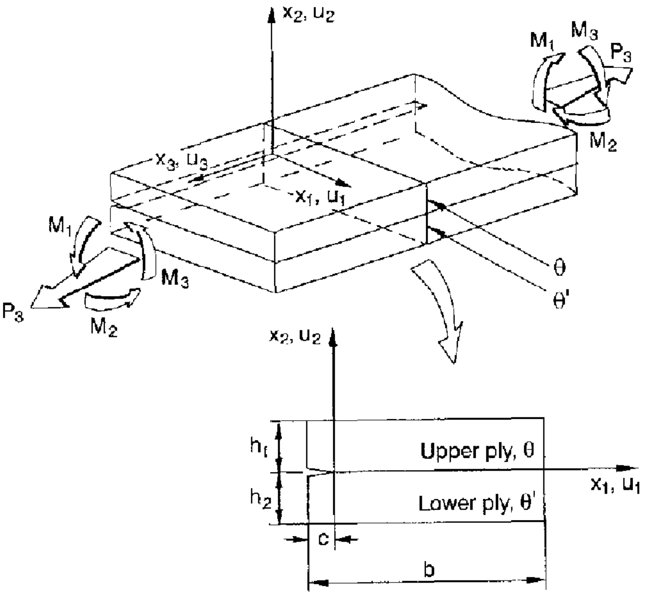

Fig. 1. A laminated composite strip with an edge delamination crack under the generic loadings of generalized plane deformations

thickness direction of the laminate (see Fig. 1). Then the x_1-axis becomes along the width direction of strip or along the ply interface of the laminate, and each ply of the composite laminate lies in a plane parallel to the x_1-x_3 plane. The ply orientation is now defined to be the counterclockwise angle, viewed from the top, that the fiber direction makes with the x_3-axis. The orientation of the adjacent plies between which a delamination crack is formed is denoted as $[\theta/\theta']$, where θ and θ' are the ply orientation of the upper and the lower ply, respectively.

The two dimensional formulation of such a class of deformations is tractable once deformation is decomposed into two parts: one is the cross-sectional distortion which is independent of the axial coordinate of strip and the other is the remaining part dependent upon the axial coordinate as well as the coordinates on the cross-section. According to Lekhnitskii (1962), the displacement u_i may indeed be written as

$$u_i(x_1,x_2,x_3) = U_i(x_1,x_2) + \delta_{i1}\left(-\frac{A_2}{2}x_3^2 - A_4x_2x_3\right)$$
$$+ \delta_{i2}\left(-\frac{A_3}{2}x_3^2 + A_4x_1x_3\right)$$
$$+ \delta_{i3}(A_2x_1 + A_3x_2 + A_1)x_3, \qquad (2.1)$$

where δ_{ij} is the Kronecker delta, and A_1 is a deformation parameter related to axial extension along the x_3-axis, A_2 and A_3 are parameters related to curvatures in the x_1-x_3 planes and the x_2-x_3 planes, respectively, and A_4 is the parameter related to twist along the x_3-axis. Let ε_{ij} and σ_{ij} denote the Cartesian components of strain and stress, respectively. For the aformentioned class of deformations, we have the following governing equations:

$$\sigma_{i\beta,\beta} = 0 \quad (i = 1,2,3, \beta = 1,2), \qquad (2.2)$$

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2, \qquad (2.3)$$

$$\sigma_{ij} = C_{ijkm}\varepsilon_{km}, \quad C_{ijkm} = C_{jikm} = C_{kmij}, \qquad (2.4)$$

where C_{ijkl} is a 4-th order stiffness tensor, and the comma indicates the partial differentiation with respect to x_i. We suppose the loading of uniaxial tension, pure bending or torsion, only, which results in the state of generalized plane deformation in the composite strip, and makes the problem two dimensional.

Substituting the expression for displacement (2.1) into the equilibrium Eq. (2.2), we obtain the following governing equation for $U(x_1, x_2)$:

$$C_{i1k1}\frac{\partial^2 U_k}{\partial x_1^2} + (C_{i1k2} + C_{i2k1})\frac{\partial^2 U_k}{\partial x_1 x_2} + C_{i2k2}\frac{\partial^2 U_k}{\partial x_2^2}$$
$$+ (C_{i123}A_4 + C_{i133}A_2) + (C_{i233}A_3 - C_{i213}A_4) = 0. \qquad (2.5)$$

In the case of the end loading being prescribed on a cross-section, the deformation A_i $(i = 1,2,3,4)$ in the above equation may be determined from the condition that the traction resultants are in equilibrium with the prescribed loadings such as the axial force, the bending moment and/or the twisting moment. Note that Eq. (2.5) with the deformation parameters A_i $(i = 1,2,3,4)$ being zero is just the governing equation for the generalized plane strain deformation. Moreover, this is an elliptic partial differential equation, characteristics of which are complex. The general solution for U_i in Eq. (2.5) is given as the sum of the homogeneous and the particular solution:

$$U_i(x_1,x_2) = U_i^h(x_1,x_2) + U_i^p(x_1,x_2). \qquad (2.6)$$

For delamination cracks, the major singularity of inverse square root appears in the homogeneous solution U_i^h. Moreover the weak singularities, such as the logarithmic singularity found in free edge, do not appear either in U_i^h or U_i^p for delamination cracks, so that the only singular terms appear in the homogeneous solution with the inverse square root singularity or the oscillatory singularity (Kim and Im (1995)). For the enriched FEM, we need to decompose a solution into a singular part and a nonsingular (regular) part; the singular part comes from the homogeneous solution. The asymptotic homogeneous solution for interfacial cracks, which includes the singular part of the solution, has been obtained in terms of the complex characteristics by Ting (1986) and Kim and Im (1995) as follows:

$$U_i^h = \sum_{n=1}^{\infty}\sum_{k=1}^{3}[\beta_n b_{kn} v_{ik} z_k^{\delta_n+1} + \beta_n b_{kn}\bar{v}_{ik}\bar{z}_k^{\delta_n+1}]/(\delta_n+1), \qquad (2.7)$$

$$\sigma_{ij}^h = \sum_{n=1}^{\infty}\sum_{k=1}^{3}[\beta_n b_{kn}\tau_{ijk} z_k^{\delta_n} + \beta_n b_{kn}\bar{\tau}_{ijk}\bar{z}_k^{\delta_n}], \qquad (2.8)$$

$$\tau_{ijk} = \sum_{m=1}^{3}(C_{ijm1} + \mu_k C_{ijm2})v_{mk} \quad (\text{no sum on } k), \qquad (2.9)$$

where β_n are real or complex free constants to be determined from the far field conditions; v_{ik} is an eigenvector for an eigenvalue μ_k for the following eigenvalue equation resulting from equilibrium Eq. (2.5):

$$[C_{i1j1} + \mu_k(C_{i1j2} + C_{i2j1}) + \mu_k^2 C_{i2j2}]\, v_{jk} = 0, \quad \text{(no sum on } k) \tag{2.10}$$

Moreover b_{kn} is the eigenvector for an eigenvalue δ_n of a characteristic equation, which is determined from appropriate near-field conditions:

$$\sigma_{2j} - \sigma'_{2j} = 0 \quad \text{and} \quad u_j - u'_j = 0 \quad \text{on} \quad x_2 = 0 \quad \text{and} \quad x_1 > 0$$

(along the ply interface),

$$\sigma_{2j} = 0 \quad \text{on} \quad x_2 = 0^+ \quad \text{and} \quad x_1 < 0$$

(along the upper crack face),

$$\sigma'_{2j} = 0 \quad \text{on} \quad x_2 = 0^- \quad \text{and} \quad x_1 < 0$$

(along the lower crack face),

where the prime "′" indicates the quantity for the lower ply.

Note that the previous equations and all relevant equations to follow, in principle, have to be written for each of the two adjacent plies at a crack, and we employ the expressions without "prime" for the upper ply and the expressions with prime for the lower ply whenever it is necessary to distinguish one from the other.

3 Formulation of enriched FEM

Crack faces of delamination are either opened or closed depending upon the ply orientations, the cross-sectional geometry and the far field loading. In this paper, we restrict our concern to an opened delamination crack, the eigenvalues δ_n in the asymptotic homogeneous solution (2.7)–(2.9) are given as (Kim and Im (1995))

$$m - \tfrac{1}{2} \pm i\eta, \quad m - \tfrac{1}{2} \quad \text{and} \quad m \quad (m = 0, 1, 2, 3, \ldots)$$

for an opened crack. (3.1)

where η is a real constant.

In the present enriched FEM we need to incorporate the singular part of solution for the displacement $U_i(x_1, x_2)$ into finite elements. The eigenvalues corresponding to $m = 0$ in Eq. (3.1) will yield the singular part of solution, and let β_1, β_2 and β_3 denote the free constants corresponding to $\delta_1 = -1/2 + i\eta$, $\delta_2 = -1/2 - i\eta$ and $\delta_3 = -1/2$, respectively. According to Eq. (2.7), the singular term may then be written as,

$$u_i^s(x_1, x_2) = \hat{u}_i^1 + \hat{u}_i^2 + \hat{u}_i^3, \tag{3.2}$$

where

$$\hat{u}_i^1 = \sum_{k=1}^{6} 2\mathrm{Re}[\beta_1]\,\mathrm{Re}\,[b_{k1} v_{ik} z_k^{1/2+i\eta}/(\tfrac{1}{2} + i\eta)], \tag{3.3}$$

$$\hat{u}_i^2 = -\sum_{k=1}^{6} 2\mathrm{Im}[\beta_1]\,\mathrm{Im}\,[b_{k1} v_{ik} z_k^{1/2+i\eta}/(\tfrac{1}{2} + i\eta)], \tag{3.4}$$

$$\hat{u}_i^3 = \sum_{k=1}^{6} \beta_3\,\mathrm{Re}\,[b_{k3} v_{ik} z_k^{1/2}/(\tfrac{1}{2})], \tag{3.5}$$

with $v_{ik+3} = \bar{v}_{ik}$, $z_{k+3} = \bar{z}_k$ $(k = 1, 2, 3)$. Note that the complex free constant β_2 corresponding to $\delta_2 = -1/2 - i\eta$ is conjugate to β_1, and its contribution is included in the above expression by doubling the contribution from β_1. The complex or real free constants β_1, β_2, β_3 are to be determined from the enriched FEM. For convenience we put

$$\beta_1 = \tfrac{1}{2}(\gamma_1 - i\gamma_2), \tag{3.6}$$

$$\beta_3 = \gamma_3. \tag{3.7}$$

Now the cross-sectional distribution $U_i(x_1, x_2)$ may be written as

$$U_i(x_1, x_2) = \sum_{a=1}^{3} \gamma_a \hat{u}_i^a(x_1, x_2) + \hat{u}_i^r(x_1, x_2), \tag{3.8}$$

where $\hat{u}^r$ represents the regular part of the displacement field and γ_a are the unknowns which have to be determined from the enriched FEM. The regular part in the expression (3.8) can be given in terms of standard interpolation:

$$\hat{u}_i^r(x_1, x_2) = \sum_{j=1}^{N} \phi_j \hat{d}_{ij}. \tag{3.9}$$

where ϕ_j and $\hat{d}_{ij}$ represent the interpolation functions and the regular nodal variables, respectively, and N is the total number of nodal points. For convenience in treating the geometric boundary conditions, it is desirable to recast the Eq. (3.8) as follows:

$$\begin{aligned} U_i(x_1, x_2) &= \sum_{a=1}^{3} \gamma_a \hat{u}_i^a + \hat{u}_i^r \\ &= \sum_{a=1}^{3} \gamma_a \left(\hat{u}_i^a - \sum_{j=1}^{N} \phi_j \hat{d}_{ij}^a \right) + \sum_{j=1}^{N} \phi_j \left(\hat{d}_{ij} + \sum_{a=1}^{3} \gamma_a \hat{d}_{ij}^a \right) \\ &= \sum_{a=1}^{3} \gamma_a u_i^a + \sum_{j=1}^{N} \phi_j d_{ij}, \end{aligned} \tag{3.10}$$

where $\hat{d}_{ij}^a$ is the nodal value of $\hat{u}_i^a$ at the node j, and

$$\begin{aligned} u_i^a &= \hat{u}_i^a - \sum_{j=1}^{N} \phi_j \hat{d}_{ij}^a, \\ d_{ij} &= \hat{d}_{ij} + \sum_{a=1}^{3} \gamma_a \hat{d}_{ij}^a. \end{aligned} \tag{3.11}$$

For convenience of representation, we rewrite Eq. (3.10) in the matrix form

$$U(x_1, x_2) = [N^r(x_1, x_2), N^s(x_1, x_2)] \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{\gamma} \end{bmatrix}, \tag{3.12}$$

where $\boldsymbol{d}$ and $\boldsymbol{\gamma}$ denote the nodal vector d_{ij} and the free constant γ_a in the vector form, and $\boldsymbol{N}^r$ and $\boldsymbol{N}^s$ are the corresponding shape functions. For simplicity of formulation we now employ the matrix representation of the stress and strain components such that $\sigma = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}]^T$ and $\varepsilon = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{23}, 2\varepsilon_{31}, 2\varepsilon_{12}]^T$. Substituting the preceding expression for $U_i(x_1, x_2)$ into Eq. (2.1), we can obtain the strain matrix by way of Eq. (2.3),

$$\varepsilon = LU + \varepsilon_0 = L\,[N^r, N^s] \begin{bmatrix} d \\ \gamma \end{bmatrix} + \varepsilon_0 = [B^r, B^s] \begin{bmatrix} d \\ \gamma \end{bmatrix} + \varepsilon_0, \quad (3.13)$$

where L is the strain operator matrix, and ε_0 is the initial strain-like term resulting from the deformation parameters A_i $(i = 1, 4)$

$$\varepsilon_0 = \begin{bmatrix} 0 \\ 0 \\ A_2 x_1 + A_3 x_2 + A_1 \\ A_4 x_1 \\ -A_4 x_2 \\ 0 \end{bmatrix}. \quad (3.14)$$

In passing we remark that the case of a prescribed uniform extension with $A_2 = A_3 = A_4 = 0$ has been treated by Wang (1984) and more recently by Chow and Atluri (1995b). However, nonzero values of A_2, A_3 and A_4 always appear even for uniform tension in the presence of elastic coupling among the plies comprising a composite laminate. The corresponding stress matrix may be obtained from Eq. (2.4):

$$\sigma = C\varepsilon = C\,[B^r, B^s] \begin{bmatrix} d \\ \gamma \end{bmatrix} + C\varepsilon_0. \quad (3.15)$$

Following the standard procedure resulting from the principle of virtual work in the absence of lateral traction, we have

$$\int_{\partial V} \sigma : \delta\varepsilon dV = 0, \quad (3.16)$$

and then we can obtain the finite element equation:

$$Kq = Q. \quad (3.17)$$

Here the stiffness matrix K, displacement vector q and the load vector Q are given as

$$K = \begin{bmatrix} K^{rr} & K^{rs} \\ K^{sr} & K^{ss} \end{bmatrix}, \quad q = \begin{bmatrix} d \\ \gamma \end{bmatrix}, \quad Q = \begin{bmatrix} Q^r \\ Q^s \end{bmatrix}, \quad (3.18)$$

and

$$K^{rr} = \int_A (B^r)^T C B^s dA, \quad K^{rs} = \int_A (B^r)^T C B^s dA = (K^{sr})^T,$$

$$K^{ss} = \int_A (B^s)^T C B^s dA, \quad (3.19)$$

$$Q^r = -\int_A (B^r)^T C\varepsilon_0 \mathrm{d}A, \quad Q^s = -\int_A (B^s)^T C\varepsilon_0 \mathrm{d}A,$$

where A indicates the cross section of the strip and, "r" and "s" indicate the regular and singular term, respectively. Note that the virtual work terms from the deformation parameters A_i in the expression (2.1) contribute to the loading term Q just like the initial strain (see Chan and Ochoa (1990)). The submatrices K^{rs} (K^{sr}) and K^{ss} contain singular terms, and therefore their computation requires very accurate numerical integration. For the present problem, we employ 24 × 24 Gaussian quadrature for computing K^{rs} and K^{ss}. Moreover the aspect ratio is maintained not far from 1 for all elements on the cross sectional domain.

4 Stress intensities and energy release rates for delamination cracks

For interface cracks such as delamination cracks, the interfacial resistance to crack growth is mode dependent in general: the growth resistance or the critical energy release rate is known to increase as the mode II becomes dominant over mode I (see Hutchinson (1991)).

Hence the J-integral or the energy release rate itself is not enough to judge how critical the state of crack tip traction field is. We need to introduce the parameters characterizing the near tip traction field in more detail. The state of three asymptotic traction components is represented by Suo's intensities (1990). Following this, for an opened interfacial crack Kim and Im (1995) wrote the asymptotic traction as:

$$t^s(r, 0) = \frac{1}{\sqrt{2\pi r}} \{K r^{i\eta} w + \bar{K} r^{-i\eta} \bar{w} + K_3 w^0\}, \quad (4.1)$$

as r approaches zero, where

$$K = \sqrt{2\pi}\beta_1, \quad K_3 = 2\sqrt{2\pi}\beta_3, \quad \eta = \mathrm{Im}(\delta_1),$$

$$w_j = \sum_{k=1}^{3} (b_{k1}\tau_{2jk} + b_{(k+3)1}\tau_{2jk}), \quad w_j^0 = \mathrm{Re}\left(\sum_{k=1}^{3} b_{k3}\tau_{2jk}\right), \quad (4.2)$$

where note that $\delta_{1,2} = -1/2 \pm i\eta$ and $\delta_3 = -1/2$ are the crack tip singularities with the imaginary part η and 0, respectively. Here t^s is the singular traction vector, and w and w^0 are the eigenvectors from an interfacial crack problem (see Appendix). For delamination cracks in composite laminates under the generalized plane deformation, the above stress intensity factors, K and K_3, that is, three real intensities $\mathrm{Re}[K]$, $\mathrm{Im}[K]$ and K_3, fully characterize the near-tip traction field for an interfacial delamination crack. We need the three scaling parameters $\mathrm{Re}[K]$, $\mathrm{Im}[K]$ and K_3 for representing the asymptotic traction state when the imaginary part of singularity η is given. However, the complex intensity factor K has the length scale dependency, and we therefore use the following stress intensity factors K_1 and K_2, as suggested by Rice (1988), based upon a specific reference length $\hat{r}$

$$t^s(r, 0) = \frac{1}{\sqrt{2\pi r}} \{(K_1 + iK_2)(r/\hat{r})^{i\eta} w + (K_1 - iK_2)(r/\hat{r})^{-i\eta} \bar{w} + K_3 w^0\}, \quad (4.3)$$

where $\hat{r}$ is a reference length (see Rice (1988)), and the definition of K_1 and K_2 is apparent from Eq. (4.1) and (4.3). As seen from Eq. (4.3), the near field traction for an opened delamination crack is not related to the three stress intensity factors in the classical sense; that is, the stress intensities are not decomposed into the three independent modes in the classical fracture mechanics, but the three modes are inherently coupled with one another. Moreover, the eigenvectors w and w^0 are dependent upon the material properties as well as upon how the coefficient vector b_{kn} is normalized.

The energy release rate introduced by Suo (1990) is

$$G = \alpha(K_1^2 + K_2^2) + \xi K_3^2, \tag{4.4}$$

where

$$\alpha = \frac{w^T(R + \bar{R})\bar{w}}{4\cosh^2 \pi\eta}, \quad \xi = \tfrac{1}{8} w^{0T}(R + \bar{R}) w^0, \tag{4.5a, b}$$

and R is a Hermitian matrix (see Appendix).

Three real scaling parameters K_1, K_2 and K_3 fully characterize the near-tip for an interfacial delamination crack when the reference length $\hat{r}$ is fixed. The relative values of the traction components along the interface or the mode mixity play an important role in the fracture or failure initiation because the crack growth resistance is mode dependent. The mode mixity will in general depend upon the three scaling parameters K_1, K_2, and K_3, for an opened interfacial crack, and moreover, it varies along the interface depending upon a ligament distance from the crack tip for a given set of K_1, K_2, and K_3. One convenient choice of defining the mode mixity may be given in terms of the traction components at the ligament distance of reference length $\hat{r}$ from the crack tip. Following this, we may define the mode mixity in terms of the phase angles Ψ_1 and Ψ_2 as

$$\Psi_1 = \tan^{-1}\left(\frac{k_2}{k_1}\right), \quad \Psi_2 = \cos^{-1}\left(\frac{k_3}{\sqrt{k_1^2 + k_2^2 + k_3^2}}\right), \tag{4.6a, b}$$

where

$$k_i = (K_1 + iK_2)\, w_i + (K_1 - iK_2)\, \bar{w}_i + K_3 w_i^0. \tag{4.7}$$

Now the three real parameters G, Ψ_1 and Ψ_2 can replace the three scalar intensity factors K_1, K_2, and K_3, and the onset of crack growth criterion may be stated as

$$G = G_c(\Psi_1, \Psi_2), \tag{4.8}$$

where G_c is the critical energy release rate, which is a material constant depending upon the mode mixity.

5
Numerical examples and discussion

For numerical examples, we choose [45/−45] and [60/−45] laminates with an edge delamination crack on its left edge under various loading within the range of generalized plane deformation (see Fig. 1). Moreover, we employ the following material data for the graphite epoxy T300/5038 (see Whitecomb (1989)),

$$E_L = 134 \text{ Gpa}, \quad E_T = E_Z = 10.2 \text{ Gpa},$$

$$G_{LT} = G_{LZ} = 5.52 \text{ Gpa}, \quad G_{TZ} = 3.43 \text{ Gpa},$$

$$\nu_{LT} = \nu_{LZ} = 0.3, \quad \nu_{TZ} = 0.49,$$

where L, T and Z indicate the principal material axes along fiber, transverse and thickness direction, respectively. Table 1 shows the eigenvalues μ_k for each of ply orientations for this material and the eigenvalues δ_s associated with delamination singularities for the two laminates.

We employ the finite element discretization with eight noded isoparametric elements as shown in Fig. 2 and it has been confirmed that such a refinement is sufficient for convergence of the present FEM solution (see Table 2). We now compute the stress intensities K_a ($a = 1, 2, 3$ for an opened crack) from the constants γ_a or β_a which is obtained from FEM solutions. Moreover, the corresponding energy release rate G is calculated via Eq. (4.4). We compare these solutions with those obtained from the method of the two-state J-integral (Jeon, Cha and Im (1995)), which has often been called the method of mutual (conservation) integral (see Chow, Beom and Atluri (1995)). Table 3 shows the comparison of the energy release rates obtained from the present method to the J-integral from Jeon, Cha and Im (1995); the J-integral was calculated via the domain integral representation in Jeon, Cha and

Table 1. The eigenvalues μ_k and stress singularities δ_s

Laminates	[45/−45]		[60/−45]	
	45	−45	60	−45
μ_1	i*0.83467	i*0.83467	i*0.75525	i*0.83467
μ_2	i*3.4719	i*3.4719	i*4.1930	i*3.4719
μ_3	i*1.1644	i*1.1644	i*1.2206	i*1.1644
δ_1	−1/2 + i*0.01049265		−1/2 + i*0.01354391	
δ_2	−1/2 − i*0.01049265		−1/2 − i*0.01354391	
δ_3	−1/2		−1/2	

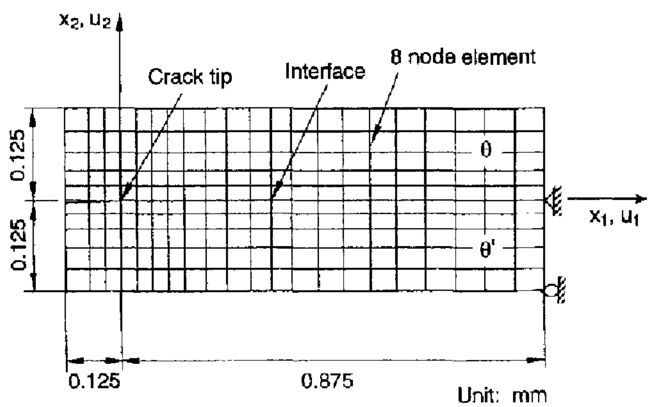

Fig. 2. A finite element mesh for the cross section of the laminates under consideration

Number of elements	[45/−45]			[60/−45]		
	K_1/P_3	K_2/P_3	K_3/P_3	K_1/P_3	K_2/P_3	K_3/P_3
64	−0.2801	1.3805	0.	−0.1922	0.9831	0.4521
176	−0.2780	1.3748	0.	−0.1914	0.9791	0.4640
336	−0.2769	1.3704	0.	−0.1911	0.9766	0.4626
396	−0.2764	1.369	0.	−0.1909	0.9759	0.4621

Table 2. Solution convergence versus the number of finite elements in terms of the stress intensities for the [45/−45] and [60/−45] composite strips under uniaxial compression ($\hat{r} = b/400$) (unit: $mm^{-3/2}$)

Laminates	Loadings R	G/R^2 (Present scheme)	J/R^2 (Two-state J-integral)
	$R = P_3 = -1$	0.1552	0.1535
[45/−45]	$R = M_1 = 1$	1.464	1.428
	$R = M_3 = -1$	10.10	10.02
	$R = P_3 = -1$	0.0924	0.0914
[60/−45]	$R = M_1 = -1$	1.130	1.101
	$R = M_3 = -1$	12.26	12.17

Table 3. Comparison of the energy release rate and the J-integral for the [45/−45] and [60/−45] laminated composite strip (unit: $N^{-1}*mm^{-1}$ or $N^{-1}*mm^{-3}$)

Laminates	Loading	Present scheme			Two-state J-integral		
	R	K_1/R	K_2/R	K_3/R	K_1/R	K_2/R	K_3/R
	$R = P_3 = -1$	−0.2764	1.369	0.	−0.272	1.353	0.
[45/−45]	$R = M_1 = 1$	0.	0.	−4.826	0.	0.	−4.749
	$R = M_3 = -1$	−7.764	−8.168	0.	−7.616	−8.139	0.
	$R = P_3 = -1$	−0.1909	0.9759	0.4621	−0.1879	0.9645	0.4578
[60/−45]	$R = M_1 = -1$	−0.9053	3.144	−1.970	−0.8831	3.093	−1.935
	$R = M_3 = -1$	−8.490	−7.325	−5.772	−8.338	−7.301	−5.745

Table 4. Comparison of the stress intensity factors from the present scheme with those from the two-state J-integral scheme (unit: $mm^{-3/2}$ or $mm^{-5/2}$)

Im (1995) while the G values are directly calculated from the free constants γ_α or β_α. The solutions for the two different schemes are in excellent agreement. Table 4 shows the stress intensities obtained from the two different approaches. Here the stress intensity factors for opened cracks are defined based upon the reference length $\hat{r} = b/400$ where b is the width of the laminate. The two solutions are in an excellent agreement again.

Figure 3 through 8 show the variation of the energy release rate and the mode mixity in terms of the phase angles versus the crack length in the [45/−45] and [60/−45] laminates subjected to the aforementioned three types of loadings. Under each of these loadings the crack faces turn out to be opened, so that our opened crack model may be applicable. Here the mode mixity in terms of the phase angles is defined based upon a fixed reference length $\hat{r}/b = 0.0025$ regardless of the crack lengths. For a wide range of change in the choice of the reference length $\hat{r}$, say the range of factor 10^4, the phase angle may change significantly, as seen in Chow and Atluri (1995a), and this fact motivated them to propose a failure criterion in terms of the complex stress intensity rather than in terms of the energy release rate and the mode mixity. However, our computation shows that the change of mode mixity depending upon the choice of the reference length $\hat{r}$ within the modest range of factor, say 10 is negligible for the delamination cracks under the present consideration. For the uniaxial compression, the energy release rate undergoes a sharp increase at an early stage of crack growth and then reaches the peak value before decreasing monotonically

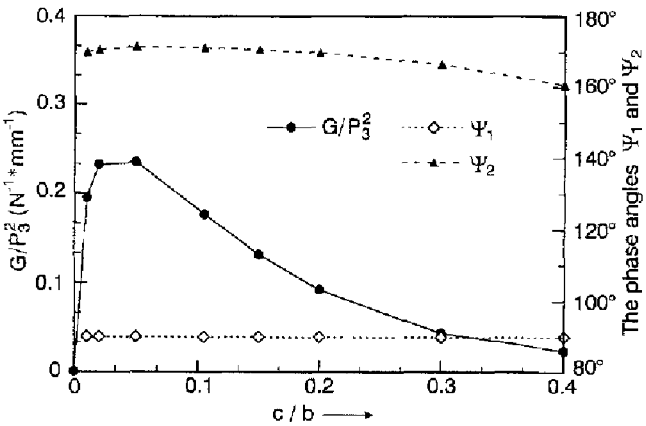

Fig. 3. The energy release rate and the mode mixity in terms of the phase angles versus the crack length for compression along the strip axis for the [45/−45] laminate

in both of [45/−45] and [60/−45] laminates. If we neglect the influence of the mode mixity change, which is not prominent near the peak value of energy release rate, there may exist a crack arrest mechanism such that the initial unstable crack growth with increasing driving force is arrested due to the decreasing driving force subsequent to the peak value. For the bending and torsion, however, the energy release rate monotonically increases as the crack grows in both of [45/−45] and [60/−45] laminates, and therefore the crack may undergo unstable growth, so that the crack growth behavior for these

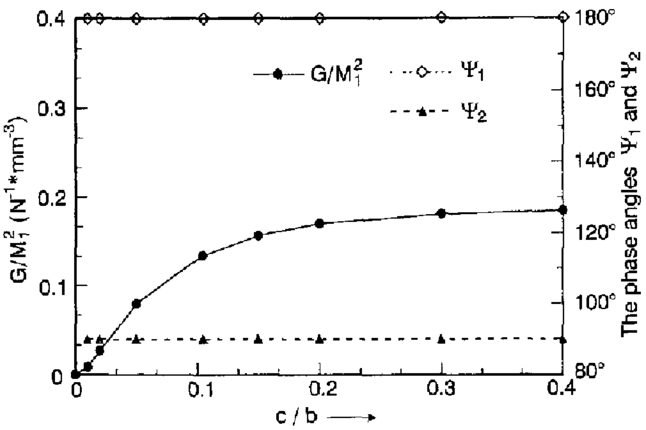

Fig. 4. The energy release rate and the mode mixity in terms of the phase angles versus the crack length of bending about the x_1-axis for the [45/−45] laminate

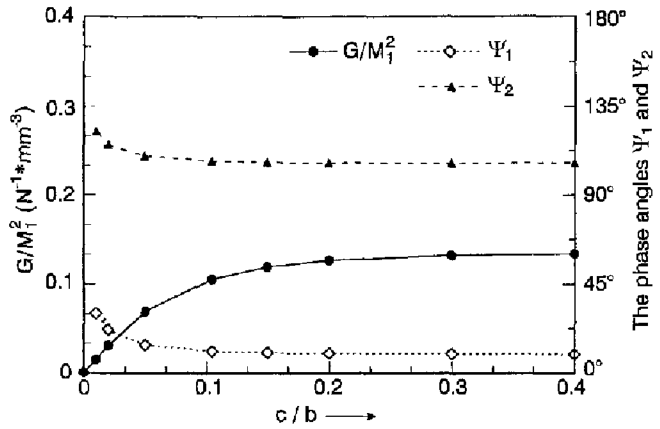

Fig. 7. The energy release rate and the mode mixity in terms of the phase angles versus the crack length for bending about the x_1-axis for the [60/−45] laminate

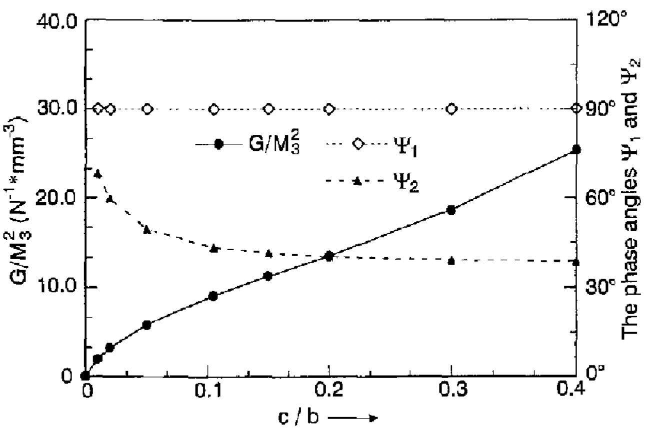

Fig. 5. The energy release rate and the mode mixity in terms of the phase angles versus the crack length for torsion about the strip axis for the [45/−45] laminate

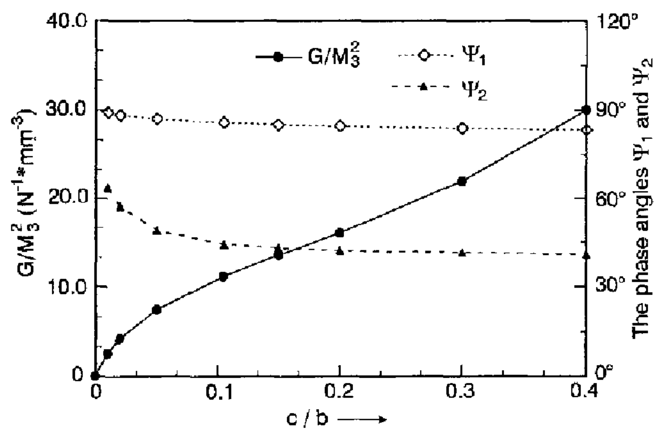

Fig. 8. The energy release rate and the mode mixity in terms of the phase angles versus the crack length for torsion about the strip axis for the [60/−45] laminate

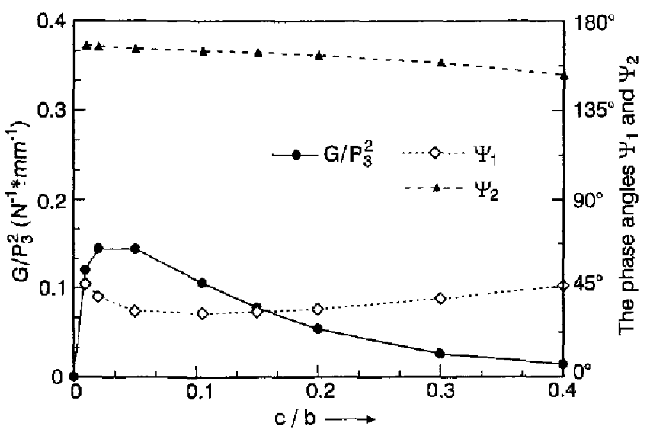

Fig. 6. The energy release rate and the mode mixity in terms of the phase angles versus the crack length for compression along the strip axis for the [60/−45] laminate

two loadings may be different from the case of the uniaxial compression. Strictly speaking, the exact growth behavior can be understood only when it is known how the resistance to crack growth depends upon the mode mixity because the mode mixity is, even if not significantly, changing as cracks grow.

Appendix

The eigenvectors w and w^0 are determined from the following eigenvalue problem obtained from the near field conditions – the traction free conditions on the crack faces and the continuity conditions along the ply interface.

$$\bar{R}w = e^{i\pi\eta}Rw, \quad \bar{R}w^0 = Rw^0, \tag{A}$$

where $R = B + \bar{B}'$, $B_{rs} = i\sum_{k=1}^{3} v_{rk}L_{ks}^{-1}$, $L_{rs} = \tau_{r2s} = \sum_{k=1}^{3}(C_{r2k1} + \mu_s C_{r2k2})v_{ks}$ (no sum on s, and $r, s = 1, 2, 3$), and w and w^0 are related to b_{kn} as in Eq. (3.3). Moreover, η is the imaginary part of the singular eigenvalues δ_1 and δ_2; μ_s and v_{ks} are explained in Sect. 2, and the prime "′" indicates the quantity for the lower ply. Kim and Im (1995) proposed a normalization of w and w^0 of $[\theta/-\theta]$ composite laminates as follows:

$$w = [*, -1/2, *]^T, \quad w^0 = [1, *, *]^T, \tag{B}$$

where (*) signifies numbers determined by Eq. (A) of the eigenvalue problem. From Eq. (3.3), it is apparent that this choice can be made by choosing appropriate normalization for the eigenvector b_{kn} in Eq. (2.6).

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